10) Applications of quasi conformal surgery
Sullivan's theorem

Let $f : \hat{\mathbb{C}} \circlearrowleft$ be a rational map of degree $d \geq 2$.

The space $\hat{\mathbb{C}}$ is divided into two totally invariant subsets, $F(f)$ and $J(f)$.

f induces a dynamical system on the connected components of $F(f)$.

In fact the image of a connected component is a open (f holomorphic) connected (continuous) subset of $F(f)$, hence it belongs to a unique connected component of $F(f)$.

Moreover if U is a connected component of the Fatou set (called Fatou component), and V is a Fatou component containing $f(U)$, then $f(U) = V$.

In fact, if $w \in \partial f(U)$, then $\exists z \in \partial U \subset J(f)$, $f(z) = w$, and $w \in J(f)$.

Def: A Fatou component $U \subset F(f)$ is called: ($f : X \circlearrowleft$)

- fixed if $f(U) = U$.
- periodic if $\exists m \geq$, $f^m(U) = U$.
- preperiodic if $\exists n > m \geq 0$ $f^n(U) = f^m(U)$ ($\Leftrightarrow$ $(f^n(U))_n$ meets only finitely many connected components of $F(f)$.
- wandering if otherwise: $(f^n(U))_n$ contains infinitely many distinct Fatou components.

Theorem (Sullivan) No wandering theorem.

For any rational map $f: \hat{\mathbb{C}} \circlearrowleft$, all Fatou components are pre-periodic. (i.e. there are no wandering Fatou components).

Rem: this theorem holds for entire maps on $\mathbb{C}$.

Example of a wandering domain: $f(z) = z + \sin(2\pi z)$.

Recently, Astorg - Buff - Dujardin - Peters - Raissy (2016) showed an example of wandering domain for a polynomial map in $\mathbb{C}^2$.

Proof based on "parabolic implosion" related to the deformation of parabolic fixed point into two distinct fixed points, which give a dynamical phenomenon called "egg beater", studied with renormalisation techniques. See the work of Lavaurs.

Other examples: $f: \mathbb{C} \circlearrowleft$ transcendent: Baker, Eremenko-Lyubich.

$f: \mathbb{C}^2 \circlearrowleft$ biholomorphic transcendent: Fornaess Sibony.

This section is devoted to the proof of Sullivan's theorem.

We start with an argument by Baker that allows to simplify the original proof by Sullivan.

First some lemmas:

Lemma: Let $f: \hat{\mathbb{C}} \circlearrowleft$ of degree $d \geq 2$. Suppose that f has a wandering Fatou component Ω. Then $\{f^n|_\Omega\}$ is Normal, and its adherence values are constants in $J(f)$ (i.e. $f^{n_k}|_\Omega \to g \Rightarrow g$ is a constant in $J(f)$)

Proof: the normality comes directly by the equivalence normality - equicontinuity, and the fact that $\Omega \subset F(f) \subset \hat{\mathbb{C}}$ compact.

Suppose $f^{n_k}|_\Omega \to g$, g non constant.

If not, $g(\Omega)$ contains a disk $D(g(z), 2\varepsilon)$ for some $\varepsilon > 0$.
Ω

Hence $f^{n_k}(\Omega) \supset D(g(z), \varepsilon)$ for $k >> 0$, and we would have

$f^{n_{k+1}}(\Omega) \cap f^{n_k}(\Omega) \neq \emptyset \Rightarrow f^{n_{k+1}}(\Omega) = f^{n_k}(\Omega)$ for $k >> 0$,

and Ω is non-wandering.

Analogously, if $g \equiv c \in F(f)$, then $c \in U$ for some Fatou component U, and $f^{n_k}(\Omega) = U$ for k big enough $\Rightarrow \Omega$ is non-wandering. $\square$

Lemma: $f: \hat{\mathbb{C}} \circlearrowleft$ is Lipschitz for the spherical metric

Proof. The spherical metric is locally defined by $\frac{|dz|}{1+|z|^2}$ (either in the chart centered at 0 with local parameter z, or at ∞ with local parameter $\frac{1}{z}$)

The lipschitzianity of f corresponds to the boundedness of $|f'(z)| \cdot \frac{1+|z|^2}{1+|f(z)|^2}$.

Locally this is bounded (up to pre or post composing by $z \mapsto \frac{1}{z}$, we may assume we are in $\mathbb{C}$, locally without poles), and it is globally bounded by compactness. $\square$

Proposition (Baker). If $f: \hat{\mathbb{C}} \circlearrowleft$, $\deg f = d \geq 2$ has a wandering Fatou component, then it has one Ω which is:

- simply connected
- $f^n: \Omega \to \Omega_n$ is a biholomorphism.

Proof: ~~Since~~ let U be a wandering (Fatou) component, and $U_n = f^n(U)$. Since $\mathcal{C}(f)$ is a finite set, we may take $N \gg 0$ so that $U_n \cap \mathcal{C}(f) = \emptyset$ $\forall n \geq N$. Set $\Omega = U_N$, $\Omega_n = f^n(\Omega)$.

Since Ω_n has no critical points, $f: \Omega_n \to \Omega_{n+1}$ defines a covering map for every $n \geq 0$.

If we show that Ω_0 is simply connected, then by the Riemann-Hurwitz formula we get $1 = \chi(\Omega_0) = e \cdot \chi(\Omega_1) \Rightarrow e = 1$, $\chi(\Omega_1) = 1$.

(Riemann-Hurwitz formula applies since f is proper).

By induction, this implies that $f|_{\Omega_n}$ is a biholomorphism and Ω_n simply connected $\forall n \geq 0$.

Up to changing coordinates we may assume that $\Omega_0 \supset \mathbb{C} \setminus \mathbb{D}$, and hence $\Omega_n \subset \mathbb{D}$ $\forall n \geq 1$.

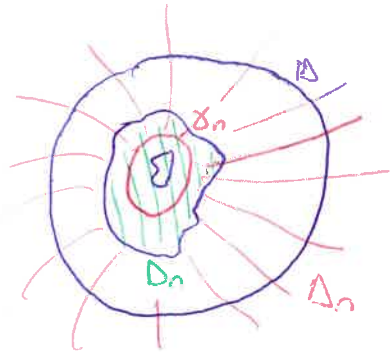
Let $\gamma_0: [0,1] \to \Omega_0$ be a non homotopically trivial loop inside Ω_0.

Denote by $a = \gamma_0(0) = \gamma_0(1) \in \Omega_0$ the base point of this loop.

The curve $\gamma_n = f^n \circ \gamma_0 : [0,1] \to \Omega_n$ is a loop in Ω_n, which is not homotopically trivial in Ω_n (or we could lift such homotopy to Ω_0 through the covering map $f^n: \Omega_0 \to \Omega_n$.

Since limit functions of $(f^n|_{\Omega_0})$ are constants we also have that $\mathrm{diam}_{\hat{\mathbb{C}}}(\gamma_n) \xrightarrow[n\to\infty]{} 0$, where $\mathrm{diam}_{\hat{\mathbb{C}}}$ is the diameter measured with the respect to the spherical metric.

Let Δ_n be the unbounded component of $\hat{\mathbb{C}} \setminus \gamma_n$. (the one containing ∞), and $D_n = \Omega_n \cup (\hat{\mathbb{C}} \setminus \Delta_n)$



Since by hypothesis γ_n is not homotopically trivial, $\hat{\mathbb{C}} \setminus \Delta_n \cap \partial\Omega_n \neq \emptyset$, and $D_n \cap J(f) \neq \emptyset$.

We claim now that there exists $m >> 0$ so that $\forall n \geq m$, $\forall C_n$ bounded connected component of $\hat{\mathbb{C}} \setminus \gamma_n$, we have $f(C_n) \subset D_{n+1}$.

Suppose the claim true: then $f(D_n) \subset D_{n+1}$, $\forall n \geq m$, and $f^k(D_m) \subset \mathbb{D}$ $\forall k \in \mathbb{N}$. But this is in contradiction with $D_n \cap J(f) \neq \emptyset$ (since in this case $\bigcup f^k(D_m) \supset \hat{\mathbb{C}} \setminus \mathcal{E}(f)$; i.e., avoids at most two points).

Let L be the Lipschitz constant of f with respect to the spheric metric, and pick $m >> 0$ so that $\mathrm{diam}_{\hat{\mathbb{C}}}(\gamma_n) \leq \frac{1}{2L}$ $\forall n \geq m$

Then: $\forall n \geq m$. (*)

$$\mathrm{diam}_{\hat{\mathbb{C}}}(f(C_n)) \leq L\, \mathrm{diam}_{\hat{\mathbb{C}}}(C_n) = L\, \mathrm{diam}_{\hat{\mathbb{C}}}(\partial C_n) \underset{(\gamma_n \supseteq \partial C_n)}{\leq} L\, \mathrm{diam}_{\hat{\mathbb{C}}}(\gamma_n) \leq \frac{1}{2}.$$

Since f is an open map, $\partial f(C_n) \subseteq f(\partial C_n) \subseteq f(\gamma_n) = \gamma_{n+1}$.

In particular $\partial f(C_n) \cap \Delta_{n+1} = \emptyset$

If $f(C_n) \not\subseteq D_{n+1}$, $\Rightarrow f(C_n) \cap \Delta_{n+1} (\cap\, \Omega_{n+1}^c) \neq \emptyset$. Since $\partial f(C_n) \cap \Delta_{n+1} = \emptyset$, this implies $\overline{\mathbb{D}}^c \subseteq \Delta_{n+1} \subseteq f(C_n)$, in contradiction with (*).

$\square$